Particles for fluids: SPH methods as a mean-field flow

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Introduction

Goal of this lecture is to show that SPH is a particle approximation of special (hydrodynamic) solutions of a kinetic equation (the Vlasov eq.n) which is a mean-field equation.
These considerations suggest approaches to show the convergence of the SPH method to the solutions of the compressible Euler eq.ns.
Similar arguments can be applied also to the Vortex method for the 2-D incompressible Euler flow.

The Vlasov equation

Physical system: A large, weakly interacting particle systems. $f(x, v, t)$ is either the fraction of particles in the cell of the phase space around (x, v) of size $dxdv$, or the probability density of finding a given particle in $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$.

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$$(\partial_t + v \cdot \nabla_x + F \cdot \nabla_v) f(x, v, t) = 0.$$

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Similar to the Liouville equation for a single particle in a force field $F = F(x, t)$. It is solved by the formula

$$f(x, v, t) = f_0(\Psi^{-t}(x, v))$$

where f_0 is the initial datum. $\Psi^t(x, v) = (x(t), v(t))$ is the flow solution to

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = F(x(t), t) \end{cases}$$

with initial datum $\Psi^0(x, v) = (x, v)$. Also, for all smooth g,:

$$\int dxdvf(x, v; t) g(x, v) = \int dxdvf_0(x, v) g(\Psi^t(x, v))$$

The Vlasov equation

F is not known a priori, but depends on the solution itself, via the selfconsistent formula

$$F(x,t) = \int dx K(x-y)\rho(y,t)dy$$

where

$$\rho(x,t) = \int dv f(x,v,t)$$

is the spatial density and $K : \mathbb{R}^d \to \mathbb{R}^d$ is a given kernel. Nonlinear equation because the vector field F depends on the solution f. K arises from a potential, namely

$$K(x) = -\nabla\varphi(x).$$

If φ is assumed to be smooth. A unique solution to the initial value problem. If $\varphi = 1/|x|$ Vlasov-Poisson.

Particle systems

Consider the N particle evolution (real physical system)

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\dfrac{1}{N} \displaystyle\sum_{\substack{j=1,N \\ j \neq i}} \nabla \varphi(x_i(t) - x_j(t)) \\ x_i(0) = x_i, \quad v_i(0) = v_i. \qquad i = 1 \dots N \end{cases}$$

Weak interaction, scaled force.

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Weak interaction, scaled force. Empirical measure defined on the one-particle phase space:

$$\mu_N(dx, dv; t) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i(t)) \delta(v - v_i(t)) dx dv.$$

Convergence

Result: If, for any smooth function g

$$\lim_{N\to\infty} \langle \mu_N(0), g\rangle = \langle f_0, g\rangle$$

for a given probability density f_0, i.e.

$$\frac{1}{N}\sum_{i=1}^{N} g(x_i(0), v_i(0)) \to \int dxdvg(x,v)f_0(x,v)$$

then

$$\lim_{N\to\infty} \langle \mu_N(t), g\rangle = \langle f(t), g\rangle = 0$$

$$\frac{1}{N}\sum_{i=1}^{N} g(x_i(t), v_i(t)) \to \int dxdvg(x,v)f(x,v;t)$$

where $f(t)$ solves Vlasov with initial datum f_0. A sort of validation of the Vlasov eq.n.

Convergence

Actually $\mu_N(t)$ **is** a solution of the Vlasov equation:

$$\frac{d}{dt}\langle \mu_N(t), g\rangle = \langle \mu_N(t), (v \cdot \nabla_x) g\rangle + \langle \mu_N(t), (F \cdot \nabla_v) g\rangle$$

$$F(x,t) = K * \mu_N(x,t) = -\frac{1}{N}\sum \nabla_x \varphi(x - x_i(t))$$

Isentropic Euler flow

Lagrangean form for the special pressure law $p = \frac{1}{2}\rho^2$.

$$\begin{cases} \ddot{\Phi}_t(x) = -\nabla\rho(\Phi_t(x), t) \\ \int \rho(x,t)g(x) = \int \rho_0(x)g(\Phi_t(x)) \\ \Phi_0(x) = x \quad \dot{\Phi}_0(x) = u_0(x), \end{cases}$$

where g is a smooth function and $\Phi_t : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is the solution flow.
The Eulerian velocity is recovered by

$$u(\Phi_t(x), t) = \dot{\Phi}_t(x).$$

Isentropic Euler flow

Next we regularize

$$\begin{cases} \ddot{\Phi}_t(x) = -\nabla(\delta_\varepsilon * \rho)(\Phi_t(x), t) \\ \int \rho(x,t) g(x) = \int \rho_0(x) g(\Phi_t(x)) \\ \Phi_0(x) = x \quad \dot{\Phi}_0(x) = u_0(x), \end{cases}$$

The form factor δ_ε is a positive, smooth approximation of the δ function. Here ε is a small positive parameter such that

$$\delta_\varepsilon \longrightarrow \delta$$

weakly, as $\varepsilon \longrightarrow 0$.

$$\langle \delta_\varepsilon, g \rangle \longrightarrow \langle \delta, g \rangle = g(0)$$

Isentropic Euler flow

The connection with the Vlasov equation. Set

$$F(x,t) = -\nabla \int \delta_\varepsilon(x-y)\rho(y,t)dy.$$

The initial (hydrodynamical) datum

$$f_0(x,v)dxdv = \rho_0(x)\delta(v-u(x)).$$

Then, setting

$$(\Phi_t(x), \dot{\Phi}_t(x)) = \Psi^t(x, u(x)),$$

the time evolved measure $f(x,v,t)dxdv$ satisfies

$$\begin{aligned}
\int f(x,v,t)g(x,v)dxdv &= \int f_0(x,v)g(\Psi^t(x,v))dxdv \\
&= \int dxdv \rho_0(x)\delta(v-u(x))g(\Psi^t(x,v)) \\
&= \int dxdv \rho_0(x)g(\Phi_t(x), \dot{\Phi}_t(x))
\end{aligned}$$

Isentropic Euler flow

In particular:

$$\int f(x, v, t) g(x) dx dv = \int \rho(x, t) g(x) dx = \int dx \rho_0(x) g(\Phi_t(x)).$$

Namely Vlasov for hydro data is the regularized Euler (Lagrangean).
The particle approximation is the SPH method.

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In particular:

$$\int f(x,v,t)g(x)dxdv = \int \rho(x,t)g(x)dx = \int dx\rho_0(x)g(\Phi_t(x)).$$

Namely Vlasov for hydro data is the regularized Euler (Lagrangean).
The particle approximation is the SPH method. Note that

$$f(x,v)dxdv = \rho(x,t)\delta(v-u(x,t)),$$

where $u(x,t)$ and $\rho(x,t)$ satisfies Euler (Eulerian) only locally in time.

SPH

The SPH model in the present context is a N-particlle system verifying

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\dfrac{1}{N} \displaystyle\sum_{\substack{j=1,N \\ j\neq i}} \nabla \delta_\varepsilon (x_i(t) - x_j(t)) \\ x_i(0) = x_i, \quad v_i(0) = u_0(x_i). \qquad i = 1 \dots N \end{cases}$$

and

$$\mu_N(dx, t) = \sum_{i=1}^{N} \delta(x - x_i(t)) dx.$$

SPH

According to the results of Theorem 1 in Section 2, we have that

$$\mu_N(dx, t) \to \rho(x, t)$$

weakly, as $N \to \infty$ for a fixed ε, where ρ is transported by the flow.

SPH

General case $\alpha \neq 0$.

$$\begin{cases} \ddot{\Phi}_t(x) = -\rho^\alpha \nabla \rho(\Phi_t(x), t) \\ \int \rho(x,t) g(x) = \int \rho_0(x) g(\Phi_t(x)) \\ \Phi_0(x) = x \quad \dot{\Phi}_0(x) = u_0(x), \end{cases}$$

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Regularized version

$$\begin{cases} \ddot{\Phi}_t(x) = -(\delta_\varepsilon * \rho)^\alpha \nabla (\delta_\varepsilon * \rho)(\Phi_t(x), t) \\ \int \rho(x,t) g(x) = \int \rho_0(x) g(\Phi_t(x)) \\ \Phi_0(x) = x \quad \dot{\Phi}_0(x) = u_0(x), \end{cases}$$

SPH

The SPH method can be suitably modified

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\frac{1}{N} \sum_{\substack{j=1,N \\ j \neq i}} (\frac{1}{N} \sum_{\substack{k=1,N \\ k \neq i}} \nabla \delta_\varepsilon (x_i(t) - x_k(t)))^\alpha \nabla \delta_\varepsilon (x_i(t) - x_j(t)) \\ x_i(0) = x_i, \quad v_i(0) = u_0(x_i). \qquad i = 1 \dots N \end{cases}$$

SPH

Results:

The method: L. Lucy (1977), J.J. Monaghan (1992).

The mean field limit for the Vlasov equation : R.L. Dobrushin (1979)

Convergence for $\alpha = 0$: K. Oelschliiger (1991), Di Lisio (1995)

For $\alpha \neq 0$ Di Lisio, Grenier, P. (1998) ....

Removing the regularization $\delta_\varepsilon \to \delta$ at level of Euler equation, namely the stability of E. eq.n w.r.t. pressure regularization: Di Lisio, Grenier, P. (1998) .

The vortex model

The Euler equation in the plane for the vorticity $\omega = \omega(x,t) = \partial_{x_1} u_2 - \partial_{x_2} u_2$ (wich is a scalar quantity in the present context) :

$$(\partial_t + u \cdot \nabla)\omega(x,t) = 0.$$

Here $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and $u = u(x,t) \in \mathbb{R}^2$. $\mathrm{div} u = 0$, implies that

$$u = \nabla^{\perp}\psi, \quad \psi = -\Delta^{-1}\omega.$$

Explicitly:

$$u = K * \omega, \qquad K(x) = \nabla^{\perp} g(x) = -\frac{1}{2\pi}\frac{x^{\perp}}{|x|^2},$$

where

$$g(x) = -\frac{1}{2\pi}\log|x|$$

is the fundamental solution for the Poisson equation in the plane.
Vlasov-like equation

The vortex model

Particle model:

$$\begin{cases} \dot{x}_i(t) = \dfrac{1}{N} \displaystyle\sum_{\substack{j=1,N \\ j \neq i}} K(x_i(t) - x_j(t)) \\ x_i(0) = x_i, \qquad i = 1 \dots N. \end{cases}$$

Smooth the singularity $K \to K_\varepsilon$ and prove convergence. Remove the singularity $\varepsilon \to 0$.
Huge literature.