The finite volume particle method for flows with moving walls

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Motivation: biomedical fluid dynamics

Mechanical heart valve
Re ≈ 6000

Bellofiore et al., 2010
The lineage of FVPM

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The finite volume particle method

Conservation law: \( \frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0 \)

Introduce a compactly supported test function \( \psi_i(x) \):

Weak form:

\[
\int_{\Omega} \psi_i \frac{\partial U}{\partial t} \, dx + \int_{\Omega} \psi_i \nabla \cdot F(U) \, dx = 0
\]

\[
\int_{\Omega} \psi_i \frac{\partial U}{\partial t} \, dx - \int_{\Omega} \nabla \psi_i \cdot F(U) \, dx = 0
\]
Choice of test function and support volume

\[ \int_{\Omega} \psi_i \frac{\partial U}{\partial t} \, dx - \int_{\Omega} \nabla \psi_i \cdot F(U) \, dx = 0 \]

\[ \psi_i(x) = \begin{cases} 
1 & x \in \Omega_i \\
0 & \text{otherwise} 
\end{cases} \]

→ finite volume method
Choice of test function and support volume

\[
\int_{\Omega} \psi_i \frac{\partial U}{\partial t} \, dx - \int_{\Omega} \nabla \psi_i \cdot F(U) \, dx = 0
\]

\[
\psi_i(x) = \frac{W_i(x)}{\sum_k W_k(x)}
\]

where \( W_i(x) = 0 \) for \( x \notin \Omega_i \)

\rightarrow \text{finite volume particle method}
Interpretation in terms of pair interactions

\[
\int_{\Omega} \psi_i \frac{\partial U}{\partial t} \, dx - \int_{\Omega} \nabla \psi_i \cdot F(U) \, dx = 0
\]

\[
\sum_{j} \frac{W_i(x) \nabla W_j(x) - W_j(x) \nabla W_i(x)}{\left( \sum_k W_k(x) \right)^2} = 0
\]

\[
\int_{\Omega} \psi_i \frac{\partial U}{\partial t} \, dx - \sum_{j} \int_{\Omega} (\gamma_{ij} - \gamma_{ji}) \cdot F(U) \, dx = 0
\]
3 approximations in FVPM, as in finite volume

1. Replace the weighted volume average of $U$ with a "particle" value

$$\frac{d}{dt} \int_{\Omega} \psi_i U d\mathbf{x} - \sum_{j} \int_{\Omega} (\gamma_{ij} - \gamma_{ji}) \cdot \mathbf{F}(U) d\mathbf{x} = 0$$

2. Represent $\mathbf{F}(U(x,t))$ with a single value for the overlap region

$$\frac{d}{dt} (V_i U_i) - \sum_{j} \beta_{ij} F_{ij} = 0$$

where

$$V_i = \int_{\Omega} \psi_i d\mathbf{x}$$

3. Reconstruct $U_i, U_j$ at the interface for the Riemann problem

$$F_{ij} \equiv F(U_i, U_j)$$
Analogy with mesh finite volume method

FVM
\[
\frac{d}{dt}(V_i U_i) - \sum_j A_{ij} \cdot (F_{ij} + \dot{x}_{ij} U_{ij}) = 0
\]

FVPM
\[
\frac{d}{dt}(V_i U_i) - \sum_j \beta_{ij} \cdot (F_{ij} + \dot{x}_{ij} U_{ij}) = 0
\]
The particle interaction vector

\[ \beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k(x) \right)^2} \, dx \]

W(x) is any kernel with compact support

2 properties of \( \beta_{ij} \)

\[ \beta_{ij} = -\beta_{ij} \quad \text{symmetry} \Rightarrow \text{exact conservation} \]

\[ \sum_j \beta_{ij} = 0 \quad \text{the particle volume is “closed”} \Rightarrow \text{zero-order consistency} \]

The mesh finite volume method is a special case of FVPM.

(Junk, 2003)
\[
\frac{d}{dt} (V_i U_i) - 2V_i \sum_j V_j G(U_i, U_j) \cdot \nabla W_i'(x_j) = 0
\]
Vila (1999)

Choose \( W' \) with double the support radius of \( W \) ⇒

\[
W_i'(x_j) = W_i(x_{ij})
\]

\[
\nabla W_i'(x_j) = \frac{1}{2} \nabla W_i(x_{ij})
\]
Relationship to ALE-SPH

Shepard-normalised RSPH kernel: \( \tilde{W}'_i(x) = \frac{W'_i(x)}{\sum_k W'_k(x)V_k} \)

Approximate relationship: \( \nabla \tilde{W}'_i(x_j) \approx \frac{1}{2} \nabla \tilde{W}_i(x_{ij}) \)

\[
\nabla \tilde{W}'_i(x_j) = \sum_j \left[ \frac{W'_i \nabla W'_j - W'_j \nabla W'_i}{\left( \sum_k W'_k \right)^2} V \right]_{x = x_j} \\
\approx \frac{1}{2} \sum_j \left[ \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k \right)^2} V \right]_{x = \bar{x}_{ij}}
\]

(if \( V_i = V_j = V \))
Relationship to ALE-SPH

ALE-SPH approximates
\[
\frac{d}{dt} (V_i U_i) - \sum_j G(U_i, U_j) \cdot V_i \left[ \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k \right)^2} \right]_{x=x_{ij}} = 0
\]

FVPM is
\[
\frac{d}{dt} (V_i U_i) - \sum_j G(U_i, U_j) \cdot \int_{\Omega_i \cap \Omega_j} \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k \right)^2} \, dx = 0
\]

overlap volume \(\cong\) material volume
\(\Rightarrow\) RSPH \(\cong\) FVPM with a single-point approximation to \(\beta_{ij}\)
A continuum from SPH to finite volume?

\[
\text{SPH} \quad -\left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) \nabla W_{ij} V_j
\]

\[
\text{ALE-SPH} \quad -2V_i V_j \nabla W_{ij} F(U_i, U_j)
\]

\[
\text{FVPM} \quad -\beta_{ij} F(U_i, U_j)
\]

finite volume \quad -A_{ij} F(U_i, U_j)
Higher-order spatial accuracy by MUSCL

- Evaluate gradients at particle barycentres using (corrected) SPH approximation

- Reconstruct $U_L$ and $U_R$ on both sides of interface

- Compute approximate numerical flux $F(U_L, U_R)$
Taylor-Green flow at Re = 100, Lagrangian
Randomised initialsation, Lagrangian
Taylor-Green, Re = 100, corrected Lagrangian
Taylor-Green flow with rogue particle
Boundary conditions

Particle support is truncated at boundary.

Compute boundary interaction vector directly…

\[ \beta_i^b = \int \frac{W_i}{\sum_k W_k(x)} \mathbf{n} d\eta \]

…or by enforcing

\[ \sum_j \beta_{ij} + \beta_i^b = 0 \]
“Complex” geometry – Reₜ = 100
SPHERIC benchmark 6: moving square

FVPM

Level set (Colagrossi)
SPHERIC benchmark 6: moving square

![Graph showing Cd vs. Time for different conditions. The graph includes lines for FVPM and REF, with different colors representing pressure and viscous effects.]
Correction of numerical $\beta_{ij}$

$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k(x) \right)^2} \, dx$$

**Numerical integration** is necessary.
Typically $6 \times 6$ quadrature points.

**Correction options**
Self-flux (Teleaga and Struckmeier, 2008)
- Preserves uniform states
- violates conservation

Pairwise shifting (Hietel and Keck, 2003)
- Restores conservation
- Errors are shifted to neighbouring particles
Computation time

- neighbour search: <1%
- flux: 4%
- gradients: 2%
- particle update: 2%
- motion correction: <1%

\[ \beta_{ij} \quad 74\% \]

- barycentres: 14%
Exact (and fast) evaluation of $\beta_{ij}$

Choose the simplest possible kernel

$$W_i(x) = \begin{cases} 1 & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k(x) \right)^2} \, d x$$

$\nabla W_i = 0$ everywhere except on boundary of $i$

Integration over $\Omega_i \cap \Omega_j$ reduces to integration along a curve
Smooth kernel functions

\[ \beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left( \sum_k W_k(x) \right)^2} \, dx \]
Top-hat kernel functions

\[ W(x) \]

\[ \Sigma W(x) \]

\[ \frac{W(x)}{\Sigma W(x)} \]
Non-overlapping top-hats = mesh finite volume
Evaluation of $\beta_{ij}$ with overlap top-hat kernel

\[
\int \frac{W_j \nabla W_i}{\left( \sum_k W_k(x) \right)^2} \, d\mathbf{x} = \int \frac{W_j \nabla W_i}{N(x)^2} \, d\mathbf{x} = \int \left( \frac{1}{N^{-}(x)} - \frac{1}{N^{+}(x)} \right) \mathbf{n} \, d\mathbf{s}
\]
Comparison of integration methods

Taylor-Green flow
Re = 100, Eulerian particles
Comparison of integration methods

Taylor-Green flow
Re = 100, nearly Lagrangian particles
Comparison of integration methods

Taylor-Green flow
Re = 100, nearly Lagrangian particles
Comparison of integration methods

Taylor-Green flow
Re = 100, nearly Lagrangian particles
Kernels for FVPM: summary
Exact $\beta_{ij}$ enables free-surface modelling

SPHERIC benchmark 5

$vectors: - \sum_j \beta_{ij}$

$t = 0.343\ s$ experiment
(Janosi et al., 2004)
Vortex-induced vibration

\[ \text{Re}_d = 100 \]
Vortex-induced vibration

\[ U_r = U_\infty / (f_n d) \]
Particle motion schemes

stationary

fixed to cylinder

interpolated

stationary

(5.5)d

7d

7d
Results
Results
Conclusions

- FVPM is closely linked to Riemann SPH
- FVPM gives robust, simple boundary treatments
- Exact interaction vectors yield $3 \times$ speedup
- Validated for bodies with prescribed and free motion

Future work

- Control of particle motion and distribution is critical
- Extension to 3D
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